Hopf Term for a Two-Dimensional Electron Gas

In a recent Letter [1], Apel and Bychkov presented microscopic calculations of the prefactor Θ of the topological Hopf term in the effective action of a two-dimensional electron gas in a magnetic field at the filling factor $\nu = 1$. They suggested that "this is the first case in condensed matter theory in which one can calculate a nonzero Hopf term from the microscopic model."

First, this is not true. A nonzero Θ was calculated microscopically for ^{3}He-A films in Ref. [2], and the results were summarized in book [3]. The method of Ref. [2] was used to microscopically calculate Θ for various lattice models in Ref. [4] and for the magnetic-field-induced spin-density-wave (FISDW) in quasi-one-dimensional organic conductors in Ref. [5]. The results of Ref. [4] were summarized in conference proceedings [6].

Second, the derivation of the Hopf term in Ref. [1] is not rigorous enough and cannot be considered as a proof. The parametrization of the rotation matrices and the unit vector $\vec{n}$ in Ref. [1] by the Euler angles $\bar{\theta}$, $\bar{\phi}$, and $\bar{\psi}$ is potentially dangerous, because the angles $\bar{\phi}$ and $\bar{\psi}$ are ill-defined where $\cos\bar{\theta} = \pm 1$. Moreover, the integrand in Eq. (14) of Ref. [1] is the total spatial derivative:

$$-\partial_x \left[\cos\bar{\theta}\, \frac{\partial(\bar{\phi},\bar{\psi})}{\partial(t,y)}\right] + \partial_y \left[\cos\bar{\theta}\, \frac{\partial(\bar{\phi},\bar{\psi})}{\partial(t,x)}\right]. \qquad (1)$$

The space integral of Eq. (1) is exactly zero, if one rotates the field $\vec{n}$ in such a way that $\cos\bar{\theta}(\infty) = 0$. Since the Hopf term is invariant under such rotation, this means that Eq. (14) of Ref. [1] does not contain the Hopf term at all. This zero result is probably an artifact of their parametrization. In the previous derivations of the Hopf term [2,4,5] the parametrization in terms of the Euler angles was avoided.

A general class of mean-field fermion models characterized by a microscopic Hamiltonian of the form

$$\hat{H} = \hat{H}_0 + \vec{\sigma}\vec{n}(\vec{r},t)\,\hat{H}_1 \qquad (2)$$

was considered in Ref. [4]. In Hamiltonian (2), which acts on the electron wave functions, $\vec{\sigma}$ are the Pauli matrices that act on the spin indices of the electrons, $\vec{n}(\vec{r},t)$ is a unit vector slowly varying in (2+1)-dimensional space-time, and the spin-independent Hamiltonians $\hat{H}_0$ and $\hat{H}_1$ are such that the system has an insulating energy gap at the Fermi level. (In the case of the BCS superconducting gap [2], the equations below are similar, but somewhat modified.) As shown in Refs. [2,4], the effective action of model (2) (obtained by integrating out the electrons) contains the Hopf term, whose coefficient Θ is given by the following expression (in the normalization of Ref. [1]):

$$\Theta = \pi N, \qquad (3)$$

where N is an integer-valued topological invariant in the momentum space (see also Ref. [7]):

$$N = \frac{1}{4\pi^2}\mathrm{Tr}\int d\omega\, dk_x\, dk_y\; G\frac{\partial G^{-1}}{\partial\omega} G\frac{\partial G^{-1}}{\partial k_x} G\frac{\partial G^{-1}}{\partial k_y}. \qquad (4)$$

In Eq. (4), k_x and k_y are the electron wave vectors in the x and y directions, ω is the Wick-rotated frequency, and

$$G(\omega,k_x,k_y) = [i\omega - \hat{H}(k_x,k_y)]^{-1} \qquad (5)$$

is the Green function of the electrons. The Hamiltonian $\hat{H}$ in Eq. (5) is given by Eq. (2) with the field $\vec{n}$ being uniform in space-time. (To derive the Hopf term, we locally transform the electrons $\psi' = \hat{U}(\vec{r},t)\psi$ to make $\vec{n}$ uniform and expand the effective action in the gradients of $\hat{U}(\vec{r},t)$ [2,4].) It is assumed that k_x and k_y are good quantum numbers, thus $\hat{H}$ and G are diagonal in k_x and k_y. The topological invariant (4) also determines the quantized Hall conductivity of the system:

$$\sigma_{xy} = Ne^2/h, \qquad (6)$$

so Θ and σ_{xy} are proportional to each other.

Thus, for a model (2), a microscopic derivation of Θ amounts to plugging the Green function of the model into Eq. (4) and doing the integral. Since the mean-field Hartree-Fock model of Ref. [1] belongs to the class (2), Eqs. (3), (4), and (6) should apply to this model. Comparing the value of the Hall conductivity at $\nu = 1$ with Eq. (6), one finds that $N = 1$, thus, from Eq. (3), $\Theta = \pi$, as suggested in Ref. [1]. Strictly speaking, integral (4) has to be somewhat modified for this model, because k_x and k_y are not good quantum numbers in the magnetic field simultaneously. That amounts, basically, to replacing the integration over the wave vectors by averaging over the phases of the boundary conditions, which is standard in the quantum Hall effect theory.

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